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REPRESENTATION OF SPINORS IN THE n — DIMENSIONAL
SPACE BY A SYSTEM OF TENSORS

by
V. A. Zhelnorovich
(USSR)

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by V. A. Zhelnorovich

1. BASIC DETERMINATIONS.

Let us consider at the outset a four-dimensional complex Euclidean space R_n^+ , $n = 2v$, referred to the orthonormalized base e_i . Let $\gamma_1, \gamma_2, \dots, \gamma_{2v}$ be the dimensionality matrices 2^v , satisfying by definition the equation

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij} I, \quad (1)$$

where δ_{ij} is the Kronecker symbol, I is the unitary dimensionality matrix 2^v . We shall introduce the denotation $\gamma_{i_1 i_2 \dots i_k} = i^{k(k-1)/2} \gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_k}$, $i_1 < i_2 < \dots < i_k$. The matrices $I, \gamma_i, \gamma_{i_1 i_2}, \dots, \gamma_{i_1 i_2 \dots i_n}$ are linearly independent and form a group of elements. As is well known, any two solutions $\gamma_i, \bar{\gamma}_i$ of Eq. (1) are linked by the equality $\bar{\gamma}_i = T \gamma_i T^{-1}$, $\det T \neq 0$, whereupon matrices γ_i may be chosen Hermitian, and in such a fashion that $\gamma_1, \gamma_2, \dots, \gamma_v$ be symmetrical, and $\gamma_{v+1}, \gamma_{v+2}, \dots, \gamma_{2v}$ antisymmetrical [1].

Let $L = \|l_p\|$ be the orthogonal transformation of the space R_{2v}^+ . The multiplicity of unimodular matrices S , determined from the equation

$$\gamma_p = l_p^q S \gamma_q S^{-1}, \quad (2)$$

form a group materializing the representation of the group L , called spinor representation.

The object $\psi = \{\psi^i\}$ with components ψ^i determined with a precision to the sign transforming according to the representation of S , is called the first rank spinor in the space R_{2v}^+ .

If γ_i is the solution of (1), it is obvious that γ_i^T (γ_i^T being the transposed γ_i) is also a solution of (1); this is why there exists a matrix C such that $\gamma_i^T = C \gamma_i C^{-1}$, $\det C = 1$.

If v is odd, it is easy to see that $C = \gamma_v \gamma_{v-1} \dots \gamma_1$; if v is even, then $C = \gamma_{2v} \dots \gamma_{v+1}$ in the case when $\gamma_1, \gamma_2, \dots, \gamma_v$ are symmetrical, but $\gamma_{v+1}, \gamma_{v+2}, \dots, \gamma_{2v}$ are asymmetrical. Hence it follows that $C^T = (-1)^{v(v-1)/2} C$. If ψ_i are the covariant components of the spinor ψ , then we have by definition $\psi_i = e_{ij} \psi^j$, where $E = \|e_{ij}\| = i^{n(n-1)/2} \gamma_n \gamma_{n-1} \dots \gamma_1 C$.

We shall outline in the space R_n^+ the pseudo-Euclidean space $R_n^{(s)}$ index \underline{s} , upon fixing the base $ie_1 \dots ie_s e_{s+1} \dots e_n$. We shall introduce the Hermitian matrix ψ determined from the equations

$$\Pi \gamma_i \Pi^{-1} = \pm \gamma_i, \quad \Pi \Pi^* = (-1)^{(v-s)(v-s+1)/2}; \quad (3)$$

where the sign minus is for $i=1, \dots, s$ and the sign plus is for $i=s+1, \dots, 2v$. The dot above the letter denotes the complex mating. The spinor $\psi = \Pi \psi$ is called conjugate relative to ψ .

Let us consider now the odd-dimensional spaces R_n^+ , $n=2v+1$. We shall denote $\gamma_{2v+1} = \gamma_{1,2,\dots,2v}$. The matrices with an even number of indices $\gamma_{i_1 i_2 \dots i_n}$ ($i_p = 1, 2, \dots, 2v+1$) are linearly independent. The spinor representation of the intrinsic orthogonal group L of space $R_{2v+1}^{(s)}$ transformation is given by the group of matrices S determined from Eq. (2), in which the indices p, q acquire values from 1 to $2v+1$. The covariant components of spinor ψ are determined by the matrix E :

$$(-1)^v \gamma_i^T = E \gamma_i E^{-1}.$$

The conjugate spinor ψ in space $R_{2v+1}^{(s)}$ is determined by the matrix Π

$$\pm \gamma_i = (-1)^{v-s} \Pi \gamma_i \Pi^{-1};$$

where the sign minus is for $i=1, 2, \dots, s$.

2. REPRESENTATION OF SPINORS BY COMPLEX TENSOR SYSTEM

Let us consider a complex matrix $\psi = \{\psi^{ij}\}$ of dimensionality \underline{r} . If $\psi^{ij} = \psi^i \psi^j$, ψ^{ij} must satisfy the equalities

$$\psi^{ij} \psi^{kl} = \psi^{ik} \psi^{jl} = \psi^{il} \psi^{kj}, \quad \psi^{ij} = \psi^{ji}, \quad (4)$$

among which $r(r+1)/2 - r$ equations $\psi^{vv} \psi^{ij} = \psi^{vi} \psi^{vj}$ ($i, j \neq v, \psi^{vv} \neq 0$) and $1/2 r(r-1)$ equations $\psi^{ij} = \psi^i \psi^j$, are independent. There are in all $(r^2 - r)$ independent equations. Reciprocally, it follows from (4) that there exists a system of \underline{r} components ψ^k , determined with a precision to signs such that $\psi^{ij} = \psi^i \psi^j$. In reality, if $\psi^{vv} = 0$ for all v , it follows from (4) that $\psi^{ij} = 0$ for all values of indices i, j . If $\psi^{vv} \neq 0$, we postulate

$$\psi^k = \psi^{vk} / \pm \sqrt{\psi^{vv}}. \quad (5)$$

By the strength of (4) such a definition of ψ^k does not depend on the value of the index v . Let us now assume that ψ^{ij} are components of the object Ψ , transforming according to the representation $S \times S$, where S is any representation of a certain group, and let Eqs. (4) be invariant relative to the group $S \times S$. Then the components ψ^k , determined according to (5) will be transformed according to the representation S . Indeed, it follows from (5) that the transformation of ψ^k is determined by the transformation of ψ^{ij} in a single fashion. Evidently, the identity (4) invariance is maintained if ψ^k is transformed according to the representation S , consequently, by the strength of the uniqueness, ψ^k may be transformed only by the representation S . Thus the object ψ^{ij} , satisfying the identities (4) is equivalent to the object ψ^k .

Let S be the spinor representation of a 2ν -dimensional orthogonal group. It is well known that

$$S \times S \sim \sum_{k=0}^{2\nu} D^k,$$

where D^k is a representation by which a tensor of rank k , asymmetrical by all indices, is transformed. This means that in this case the object ψ^{ij} is equivalent to the tensor aggregate $\Lambda = \{c_0, c_i, c_{i_1 i_2}, \dots, c_{i_1 i_2 \dots i_{2\nu}}\}$. If S is the spinor representation of a $(2\nu + 1)$ -dimensional group

$$S \times S \sim \sum_{k=0}^{\nu} D^{2k},$$

and consequently, the object ψ^{ij} is equivalent to the tensor aggregate consisting of even rank tensors. The components of the antisymmetrical tensors $c_{i_1 i_2 \dots i_k}$ may be determined as follows:

$$c_{i_1 i_2 \dots i_k} = (A_{i_1 i_2 \dots i_k})_{\alpha} \psi^{\alpha}, \quad A_{i_1 i_2 \dots i_k} = E \gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_k}. \quad (6)$$

Inasmuch as $\det E \neq 0$ and $\gamma_{i_1 i_2 \dots i_k}$ are linearly independent, $A_{i_1 i_2 \dots i_k}$ are also linearly independent and consequently, the aggregates Λ are really equivalent to objects ψ^{ij} .

Taking advantage of symmetry properties of C , it may be shown that matrices $A_{i_1 i_2 \dots i_k}$ have the following symmetry properties:

$$(A_{i_1 i_2 \dots i_k})^T = (-1)^{(\nu(\nu+1) + k(k+1))/2} A_{i_1 i_2 \dots i_k}.$$

This is why, so long as ψ^{ij} satisfies the identities (4), part of tensors $c_{i_1 i_2 \dots i_k}$ ($[\nu(\nu+1) + k(k+1)]/2$ being odd) become zero. If ψ^{ij} satisfy the identities (4), tensors $c_{i_1 i_2 \dots i_k}$ satisfy $1/2 2^{\nu}(2^{\nu}-1)$ independent bilinear identities, all of which are included in the generalization of the Pauli identity for the case of a n -dimensional space, (See [2])

$$\begin{aligned} 2^{\nu} (\psi^+ \theta \psi) (\psi^+ \theta' \psi) &= \sum_{k=1}^{2\nu} \sum_{i_1 < i_2 < \dots < i_k} (\psi^+ \gamma_{i_1 i_2 \dots i_k} \psi) (\psi^+ \theta' \gamma_{i_1 i_2 \dots i_k} \theta \psi) + \\ &+ (\psi^+ \psi) (\psi^+ \theta' \theta \psi), \quad n = 2\nu. \\ 2^{\nu} (\psi^+ \psi) (\psi^+ \theta' \psi) &= \sum_{k=1}^{2\nu} \sum_{i_1 < i_2 < \dots < i_k} (\psi^+ \gamma_{i_1 i_2 \dots i_{2k}} \psi) (\psi^+ \theta' \gamma_{i_1 i_2 \dots i_{2k}} \theta \psi) + \\ &+ (\psi^+ \psi) (\psi^+ \theta' \theta \psi), \quad n = 2\nu + 1, \end{aligned} \quad (7)$$

where θ', θ are arbitrary matrices of dimensionality 2^{ν} ; ψ^+, ψ are covariant and contravariant component of the spinor.

Therefore, spinor ψ^k in the space R_n^+ , $n = 2\nu, 2\nu + 1$ is equivalent to the tensor aggregate Λ , consisting of complex antisymmetrical tensors satisfying $1/2 2^{\nu}(2^{\nu}-1)$ bilinear identities (7). By virtue of this, any spinor equation may be written in an equivalent manner as an equation in tensor components $c_{i_1 i_2 \dots i_k}$.

Note that formula (3) determines the components ψ^k in any system of coordinates, but the transformation of components ψ^k to curvilinear coordinates is found to be nonlinear relative to ψ^k .

3. REPRESENTATION OF SPINORS BY A SYSTEM OF REAL TENSORS

We shall consider an r -dimensional complex matrix ψ^{pq} . Assume that $\psi^{pq} = \psi^{p\bar{q}}$.

Then ψ^{pq} will satisfy the identities

$$\psi^{pq} = (\psi^{qp})^*, \quad \psi^{mn} \psi^{pq} = \psi^{mq} \psi^{pn} \quad (8)$$

among which $(r-1)$ real equations $\psi^{vv} \psi^{pq} = \psi^{vq} \psi^{pv}$ ($p, q \neq v, \psi^{vv} \neq 0$) and r^2 real equations $\psi^{pq} = (\psi^{qp})^*$ are independent. Obviously, if the components ψ^k determine the matrix ψ^{pq} the components $\psi^k e^{i\varphi}$, and only they, determine the same matrix ψ^{pq} .

Reciprocally, it follows from (8) that there exists a system of component determined with a precision to the phase $e^{i\varphi}$, such that $\psi^{pq} = \psi^p \psi^q$. In reality if $\psi^{vv} = 0$ for all v , it will follow from (8) that $\psi^{pq} = 0$ for all p, q . In this case we postulate $\psi^k = 0$. If $\psi^{vv} \neq 0$, we postulate

$$\psi^k = \frac{\psi^{vk}}{\pm \sqrt{\psi^{vv}}} e^{i\varphi}, \quad (9)$$

where φ is an arbitrary real number.

By virtue of (8) such a determination of ψ^k multiplicity does not depend on the value of v . It may be shown that only the components ψ^k , determined according to (9) satisfy the equation $\psi^{pq} = \psi^p \psi^q$.

Assume that ψ^{pq} are components of an object transforming according to the representation $S \times S$, where S is any representation of a certain group, and let the equalities (8) be invariant relative to the group $S \times S$. Then, we may evidently point to such a law of transformation of φ that the components ψ^k transform according to the representation S .

In this way the assignment of the object ψ^{pq} satisfying the identities (8) and of the argument φ of one of the components ψ^k fully determine the object. Let S be the spinor representation of a $2v$ -dimensional orthogonal group of transformations of the space $R_{2v}^{(e)}$. It is well known that

$$S \times S \sim \sum_{k=0}^{2v} D^k.$$

This means that the object ψ^{pq} is equivalent to the tensor aggregate $\Omega = \{\Omega_0 \Omega_{i_1} \dots \Omega_{i_{i_2}} \dots i_{2v}\}$, consisting of antisymmetrical tensors.

If S is a spinor representation in the space $R_{2v+1}^{(e)}$, $S \times S \sim \sum_{k=0}^v D^{2k}$, and in this case the object ψ^{pq} is equivalent to the aggregate $\Omega = \{\Omega_0 \Omega_{i_1 i_2} \dots \Omega_{i_1 i_2 \dots i_{2v}}\}$, consisting of antisymmetrical tensor of even rank.

The components of these tensors may be determined as follows:

$$\Omega_{i_1 i_2 \dots i_k} = (D_{i_1 i_2 \dots i_k})_{\alpha\beta} \psi^{\alpha\beta}, \quad D_{i_1 i_2 \dots i_k} = E \Pi \gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_k} i^{k(k+1)/2}.$$

Utilizing (3), we may show that matrices $D_{i_1 i_2 \dots i_k}$ are Hermitian.

This is why, the components $\Omega_{i_1 i_2 \dots i_k}$ are real if ψ^{pq} satisfy the identity $\psi^{pq} = (\psi^{qp})^*$. If ψ^{pq} satisfy the identities (8), the components of tensors $\Omega_{i_1 i_2 \dots i_k}$ satisfy $(2^v - 1)^2$ bilinear identities, of which everyone is contained in the identity (7).

Therefore, the assignment of the aggregate Ω and of argument φ of one of the components fully determines the spinor. This means that the spinor equations

may be written in an equivalent manner in components of the aggregate Ω and φ . Then, eliminating from such equations the argument φ , it is possible to obtain a closed system of equations in components of aggregate Ω .

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Moscow State University in the name of
M. V. Lomonosov

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1145, 19th st NW
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1. J. A. SHOUTEN. Indag. Math, 11, 3, 4, 5 (1949)
2. K. M. CASE. Phys. Rev., 97, 3, 810, 1955.

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